Indian Statistical Institute, Bangalore

B. Math.(Hons.) I Year, Second Semester

Mid-Sem Examination

Analysis -II

Time: 3 hours March 1, 2010 Instructor: Pl.Muthuramalingam

Maximum Marks 40

1. a) Let $a_i, b_i, c_i \geq 0$ for $i = 1, 2$ with $c_i \leq a_i + b_i$. Show that $(c_1^2 + c_2^2)^{1/2} \leq (a_1^2 + a_2^2)^{1/2} + (b_1^2 + b_2^2)^{1/2}$. [3]

   b) Let $a, b, c \geq 0$ with $c \leq a + b$ if $c_1 = \min\{1, c\}, b_1 = \min\{b, 1\}, a_1 = \min\{a, 1\}$ show that $c_1 \leq a_1 + b_1$. [1]

   c) Let a, b, c be as in (b). Show that

   $$\frac{c}{1+c} \leq \frac{a}{1+a} + \frac{b}{1+b}.$$

   [1]

   d) Let $a_1, a_2, \cdot$ be a sequence with $a_n \geq 0$. Show that $a_n \longrightarrow 0$ iff min $\{a_n, 1\} \longrightarrow 0$. [1]

   e) Let a_i be as above. Then $a_n \longrightarrow 0$ iff $\frac{a_n}{1+a_n} \longrightarrow 0$. [1]

2. a) Let (X, d) be disconnected metric space. Show that there exists a continuous, on to function $g : (X, d) \longrightarrow \{o, 1\}$. [2]

   b) Let (Y, m) be any metric space such that there exists a continuous, onto function $f : (Y, m) \longrightarrow \{0, 1\}$. then (Y, m) is disconnected. [1]

   c) Let $(X, d), (Y, m)$ be connected metric spaces. $Z = X \times Y$. Let p be the metric on Z given by $p((x_1, y_1), (x_2, y_2)) = \{[d(x_1, x_2)]^2 + [m(y_1, y_2)]^2\}^{1/2}$. Show that (Z, p) is connected. [3]

   d) A continuous image of a connected space is connected. [2]

   e) If $\{c_\alpha : \alpha \epsilon I\}$ is a (finite or infinite) family of connected subsets in (X, d) such that $\bigcap c_\alpha \neq$ empty, then $\bigcup c_\alpha$ is a connected subset of (X, d). [2]

3. a) Let C be any countable infinite subset of R^2. Then show that $R^2 - C$ is connected. [4]

   b) Let (X, d) be a metric space. Let $A \subset B$ with A connected and for each b in B there exists a sequence $a_1, a_2, \cdots$ in A such that $d(a_n, b) \longrightarrow 0$. Show that B is connected. [2]

   c) Show that $A = \{(x, y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1\}$ is a connected subset of R^2. [2]

d) Let S be any subset of $[0, 2\pi]$. Show that with A as in (c) the set $A \bigcup \{(a\cos\theta, b\sin\theta) : \theta \text{ in } S\}$. is a connected subset of R^2. [1]

4. a) If A is any bounded closed subset of R^r, with usual metric, show that A has Bolzano Weistrass(BW) property. [3]

   b) If (X, d) has BW and $f : (X, d) \longrightarrow R$ is continuous, then f is a bounded function. [2]

5. Let $f(x_1, x_2) = x_1\, x_2[x_1^4 + x_2^4]$ calculate $f(0,0), (\nabla f)(0,0), ((\frac{\partial^2 f}{\partial x_1 \partial x_j}))(0,0)$. Show that $(0,0)$ a saddle point for f. [2]

6. Let Q be the quadratic form on R^3 given by $Q(x_1, x_2, x_3) \equiv x_1^2 + 3x_2^2 + x_3^2 - 4x_1\, x_2 - 2x_1\, x_3 - 6x_2\, x_3$. Determine if $Q > 0$ or $Q < 0$ or neither.

   [3]

7. You are to construct an open rectangular box with a given volume V. Of all such boxes find one with minimum surface area by using Lagranges method. [4]